Gauge theories on noncommutative $\mathbb{C}P^N$ and BPS-like equations

[1] Akifumi Sako, [2] Toshiya Suzuki and [2] Hiroshi Umetsu

[1] Department of Mathematics, Faculty of Science Division II,
Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
[2] National Institute of Technology, Kushiro College
2-32-1 Otanoshike-nishi, Kushiro, Hokkaido 084-0916, Japan

MSC 2010: 53D55 , 81R60

Abstract

We give the Fock representation of a noncommutative $\mathbb{C}P^N$ and gauge theories on it. The Fock representation is constructed based on star products given by deformation quantization with separation of variables and operators which act on states in the Fock space are explicitly described by functions of inhomogeneous coordinates on $\mathbb{C}P^N$. Using the Fock representation, we are able to discuss the positivity of Yang-Mills type actions and the minimal action principle. Other types of actions including the Chern-Simons term are also investigated. BPS-like equations on noncommutative $\mathbb{C}P^1$ and $\mathbb{C}P^2$ are derived from these actions. There are analogies between BPS-like equations on $\mathbb{C}P^1$ and monopole equations on $\mathbb{R}^3$, and BPS-like equations on $\mathbb{C}P^2$ and instanton equations on $\mathbb{R}^8$. We discuss solutions of these BPS-like equations.

1 Introduction

We come across field theories on noncommutative spaces in various situations. For example, effective theories of D-branes with background B fields are given as gauge theories on noncommutative manifolds [20]. Another example is the IIB matrix model [8], some classical solutions of which correspond to noncommutative gauge theories. These facts have motivated analyses of field theories on noncommutative spaces. (See, for example, review papers [16, 21, 17].) In particular, it has become increasingly important to investigate properties of gauge theories on various noncommutative manifolds.

In our preceding paper [18] we provided explicit expressions of star products of noncommutative $\mathbb{C}P^N$ and $\mathbb{C}H^N$, and in [14] gauge theories on noncommutative

homogeneous Kähler manifolds are constructed by using the deformation quantization with separation of variables [10]. The aim of this article is to investigate them in some details.

Here, we briefly review the deformation quantization that is a way to realize noncommutative manifolds. It is defined as follows. Let $\mathcal{F}$ be a set of formal power series in $\hbar$ with coefficients of C^∞ functions on a Poisson manifold M i.e. $\mathcal{F} := \left\{ f \;\middle|\; f = \sum_k \hbar^k f_k,\ f_k \in C^\infty(M) \right\}$, where $\hbar$ is a noncommutative parameter. A star product is defined on $\mathcal{F}$ by

$$f * g = \sum_k \hbar^k C_k(f,g), \qquad f,g \in \mathcal{F}, \tag{1.1}$$

such that the product satisfies the following conditions.

1. $*$ is an associative product.
2. C_k is a bidifferential operator.
3. C_0 and C_1 are defined as

$$C_0(f,g) = fg, \tag{1.2}$$
$$C_1(f,g) - C_1(g,f) = i\{f,g\}, \tag{1.3}$$

   where $\{f,g\}$ is the Poisson bracket.
4. $f * 1 = 1 * f = f$.

In [14], noncommutative gauge theories on homogeneous Kähler manifolds are constructed by using deformation quantization with separation of variables. The deformation quantization with separation of variables is one of the methods to construct noncommutative Kähler manifolds given by Kalabegov [10]. (See also [9, 11].) Physical quantities like gauge fields are given as formal power series in a noncommutative parameter in deformation quantization, and therefore it is difficult to discuss the positivity and boundedness of physical quantities in general. To justify processes of the minimal action principle and deriving BPS-like equations we have to get rid of difficulties resulting from using formal power series. One of the way to do this is to choose an appropriate representation of the noncommutative algebra on $(\mathcal{F},\ *)$. For example, it is well-known that the Fock representation is a good representation of noncommutative algebra in the Moyal $\mathbb{R}^N$. For noncommutative $\mathbb{C}P^N$ the Fock representation described by using star products of Karabegov's deformation quantization is given in [18, 19].

In this article, we use the Fock representation given in [18, 19] to construct gauge theories. By virtue of the Fock representation, we are able to prove the positivity of Yang-Mills type actions and derive equations of motion. We see analogies between

these gauge theories on noncommutative $\mathbb{C}P^N$ and the gauge theories on noncommutative $\mathbb{R}^{N^2+2N}$. Based on this observation, we propose BPS-like equations for gauge theories on noncommutative $\mathbb{C}P^1$ and $\mathbb{C}P^2$. For the Yang-Mills-Higgs type model on noncommutative $\mathbb{C}P^1$, obtained BPS-like equations are similar to the monopole equations given in the gauge-Higgs model in $\mathbb{R}^3$ [2, 5]. For the Yang-Mills type theory on noncommutative $\mathbb{C}P^2$, obtained BPS-like equations are analogous to the instanton equations in $\mathbb{R}^8$ [4]. We also discuss BPS-like equations in a gauge theory on $\mathbb{C}P^2$ with an action which is a combination of the Yang-Mills type action and the Chern-Simons type action. Further, we study some solutions for these new BPS-like equations.

The organization of this paper is as follows. In Section 2, we summarize preliminaries to investigate noncommutative gauge theories. In Section 3, we reformulate the Fock representation of $\mathbb{C}P^N$ in more sophisticated manner than in [18, 19]. Using this Fock representation, we construct noncommutative gauge theories and prove the positivity of the action functional of the gauge theories. In Section 4, we derive the equations of motions, the Bianchi identities and BPS-like equations. Summaries are given in Section 5.

2 Preliminaries to gauge theories on noncommutative $\mathbb{C}P^N$

In [14], gauge theories on noncommutative homogeneous Kähler manifolds $M = \mathcal{G}/\mathcal{H}$ are constructed. In the theories, the Kähler manifolds are deformed by using deformation quantization with separation of variables given by Karabegov [10]. Here, we denote deformation quantization with separation of variables when star products satisfy

$$a * f = af, \qquad f * b = fb, \tag{2.1}$$

for any holomorphic function a and any anti-holomorphic function b.

$\mathbb{C}P^N$ is one of the typical homogeneous Kähler manifolds. In the inhomogeneous coordinates z^i $(i = 1, 2, \cdots, N)$, the Kähler potential of $\mathbb{C}P^N$ is given by

$$\Phi = \ln\left(1 + |z|^2\right), \tag{2.2}$$

where $|z|^2 = \sum_{k=1}^{N} z^k \bar{z}^k$. The metric $(g_{i\bar{j}})$ is

$$ds^2 = 2g_{i\bar{j}} dz^i d\bar{z}^j, \tag{2.3}$$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi = \frac{(1 + |z|^2)\delta_{ij} - z^j \bar{z}^i}{(1 + |z|^2)^2}, \tag{2.4}$$

and the inverse of the metric $(g^{\bar{i}j})$ is

$$g^{\bar{i}j} = (1+|z|^2)\left(\delta_{ij} + z^j \bar{z}^i\right). \tag{2.5}$$

A star product in $\mathbb{C}P^N$ is given as follows [18, 19]:

$$f * g = \sum_{n=0}^{\infty} c_n(\hbar) g_{j_1\bar{k}_1} \cdots g_{j_n\bar{k}_n} \left(D^{j_1} \cdots D^{j_n} f\right) D^{\bar{k}_1} \cdots D^{\bar{k}_n} g, \tag{2.6}$$

where

$$c_n(\hbar) = \frac{\Gamma(1-n+1/\hbar)}{n!\Gamma(1+1/\hbar)}, \qquad D^{\bar{i}} = g^{\bar{i}j}\partial_j, \qquad D^i = g^{i\bar{j}}\partial_{\bar{j}}. \tag{2.7}$$

This star product is constructed by using the way of the deformation quantization given in [10]. Note that the complex conjugate of $f*g$ is given as $\overline{f*g} = \bar{g}*\bar{f}$. It is easily found that this star product satisfies the condition of deformation quantization with separation of variables (2.1).

It should be noted that first order differential operators in noncommutative spaces do not satisfy the Leibniz rule in general, but a linear differential operator defined by using a commutator such that $\mathcal{L}(f) = [P,f]_* := P*f - f*P$, $(P, f \in C^\infty(M)[[\hbar]])$ satisfies the Leibniz rule. The star commutator $[P,f]_*$ includes higher derivative terms of f for a generic P. We have to use first order differential operators as derivations to construct field theories having usual kinetic terms. In the noncommutative Kähler manifolds deformed by deformation quantization with separation of variables, it is known that vector fields are inner derivations if and only if the vector fields are the Killing vector fields [15, 14]. Hence, field theories having usual kinetic terms should be constructed by using the star commutators with the Killing potentials corresponding to the Killing vectors. We denote the Killing vector field by $\mathcal{L}_a = \zeta_a^i \partial_i + \zeta_a^{\bar{i}}\partial_{\bar{i}}$. Note that "$a$" is not an index of tangent vector space of M but one of the Lie algebra of the isometry group $\mathcal{G}$. The Killing vector $\mathcal{L}_a$'s satisfy

$$[\mathcal{L}_a, \mathcal{L}_b] = i f_{abc}\mathcal{L}_c, \tag{2.8}$$

where f_{abc} is a structure constant of the Lie algebra of $\mathcal{G}$.

Let us review the construction of a $U(k)$ gauge theory by using the Killing vector field $\mathcal{L}_a$ [14]. The Yang-Mills type action is given as follows. We define a local gauge field $\mathcal{A}_a \in C^\infty(M)[[\hbar]] \otimes u(k,\mathbb{C})$ as a formal power series

$$\mathcal{A}_a := \sum_{j=0}^{\infty} \hbar^j \mathcal{A}_a^{(j)}, \tag{2.9}$$

and we define its gauge transformation by

$$\mathcal{A}_a \to \mathcal{A}'_a = iU^{-1} * \mathcal{L}_a U + U^{-1} * \mathcal{A}_a * U, \tag{2.10}$$

where U and U^{-1} are elements of $M_k(C^\infty(M)[[\hbar]])$. Note that the gauge field is defined not on the tangent space on M but on the tangent space on the isometry group $\mathcal{G}$ restricted into M. (For the case of $\mathbb{C}P^N$ the isometry group $\mathcal{G}$ is $SU(N+1)$.) Here we put the condition

$$U^\dagger * U = \sum_{n=0}^{\infty} \hbar^n \sum_{m=0}^{n} U^{(m)\dagger} * U^{(n-m)} = I, \tag{2.11}$$

for $U = \sum_{j=0}^{\infty} \hbar^j U^{(j)}$. The leading term $U^{(0)}$ is a map to $U(k)$. We denote $\mathcal{U}(k)$ as a set of U satisfying (2.11). Let us define a curvature of $\mathcal{A}_a$ by

$$\mathcal{F}_{ab} := \mathcal{L}_a \mathcal{A}_b - \mathcal{L}_b \mathcal{A}_a - i[\mathcal{A}_a, \mathcal{A}_b]_* - i f_{abc} \mathcal{A}_c. \tag{2.12}$$

This $\mathcal{F}_{ab}$ transforms covariantly:

$$\mathcal{F}_{ab} \to \mathcal{F}'_{ab} = U^{-1} * \mathcal{F}_{ab} * U. \tag{2.13}$$

Therefore, we obtain the Yang-Mills type gauge invariant action;

$$S_g := \int_{\mathcal{G}/\mathcal{H}} \mu_g \, \frac{1}{4} \mathrm{tr} \left(\eta^{ac} \eta^{bd} \mathcal{F}_{ab} * \mathcal{F}_{cd} \right), \tag{2.14}$$

where η is the Killing form of $\mathcal{G}$, and μ_g is a trace density i.e. $\int_M (f * g)\mu_g = \int_M (g * f)\mu_g$. For $\mathbb{C}P^N$, the trace density is given by the Riemannian volume element $\sqrt{g}$. This gauge theory is expected to be the unique gauge theory that its kinetic term contains no higher derivative and it connects to the usual Yang-Mills type theory in the commutative limit.

3 The Fock representation of noncommutative $\mathbb{C}P^N$

The Fock representation of noncommutative $\mathbb{C}P^N$ with the star product (2.6) was introduced in [18]. In this section, we improve its formulation and provide a more detailed description of it.

Under the star product (2.6), z^i and $\partial_j\Phi = z^{\bar{j}}/(1+|z|^2)$ satisfy the commutation relations for the creation-annihilation operators,

$$\left[\partial_i\Phi,\ z^j\right]_* = \hbar\delta_{ij}, \qquad \left[z^i,\ z^j\right]_* = 0, \qquad \left[\partial_i\Phi,\ \partial_j\Phi\right]_* = 0, \tag{3.1}$$

and thus it would be natural to consider that functions on the noncommutative $\mathbb{C}P^N$ are constructed from z^i and $\partial_j\Phi$. But, z^i and $\partial_j\Phi = z^{\bar{j}}/(1+|z|^2)$ are not hermitian conjugate each other, when an inner product between functions F, G is defined by

$$(F,G) = \int d^N z d^N \bar{z} \sqrt{g}\ (\bar{F} * G), \tag{3.2}$$

where $\sqrt{g} = 1/(1+|z|^2)^{N+1}$. Hence, we introduce another set of creation and annihilation operators, a_i and $a_i^\dagger$ $(i = 1, 2, \dots, N)$ which are hermitian conjugate each other, as follows;

$$a_i = \frac{1}{\sqrt{\hbar}}\partial_i\Phi * (1 - \tilde{n} + \hbar)_*^{-1/2} = \frac{1}{\sqrt{\hbar}}(1-\tilde{n})_*^{-1/2} * \partial_i\Phi, \tag{3.3}$$

$$a_i^\dagger = \frac{1}{\sqrt{\hbar}}(1 - \tilde{n} + \hbar)_*^{1/2} * z^i = \frac{1}{\sqrt{\hbar}} z^i * (1-\tilde{n})_*^{1/2}, \tag{3.4}$$

where

$$\tilde{n} = z^i * \partial_i\Phi = \frac{|z|^2}{1+|z|^2}, \tag{3.5}$$

$$[\tilde{n},\ z^i]_* = \hbar z^i, \qquad [\tilde{n},\ \partial_i\Phi]_* = -\hbar\partial_i\Phi, \tag{3.6}$$

and $f_*^{1/2} \in \mathcal{F}$ is defined by

$$f_*^{1/2} * f_*^{1/2} = f. \tag{3.7}$$

The existence of $f_*^{1/2}$ for $^\forall f \in \mathcal{F}$ is confirmed by solving (3.7) recursively. Similarly, f_*^{-1} and $f_*^{-1/2}$ are defined as

$$f_*^{-1} * f = f * f_*^{-1} = 1, \tag{3.8}$$

$$f_*^{-1/2} * f_*^{1/2} = f_*^{1/2} * f_*^{-1/2} = 1. \tag{3.9}$$

It can be easily seen that a_i and $a_i^\dagger$ satisfy the commutation relations,

$$\left[a_i,\ a_j^\dagger\right]_* = \delta_{ij}, \qquad [a_i,\ a_j]_* = 0, \qquad \left[a_i^\dagger,\ a_j^\dagger\right]_* = 0. \tag{3.10}$$

We can find that a_i and $a_i^\dagger$ are hermitian conjugates of each other from the following relations which are calculated by using the definition of star product (2.6),

$$\bar{z}^i * (1-\tilde{n}) = \bar{z}^i * \frac{1}{1+|z|^2} = (1-\tilde{n}-\hbar) * \bar{z}^i = \partial_i \Phi - \hbar \bar{z}^i, \tag{3.11}$$

$$\bar{z}^i = \partial_i \Phi * (1-\tilde{n}+\hbar)^{-1} = (1-\tilde{n})^{-1} * \partial_i \Phi. \tag{3.12}$$

The number operator n is defined as

$$n = a_i^\dagger * a_i = \frac{1}{\hbar}\tilde{n}. \tag{3.13}$$

As described in Section 2, commutators with the Killing potentials provide derivations corresponding to the Killing vector fields. $\mathbb{C}P^N$ has the $SU(N+1)$ isometry and the Killing potentials corresponding to the Killing vector fields, $\mathcal{L}_a = -\frac{i}{\hbar}[P_a, \ \ ]_*$ $(a = 1, 2, \ldots, N^2+2N)$, are given by

$$P_a = i\left[(T_a)_{00}(z^i * \partial_i \Phi - 1) - i(T_a)_{0i}\partial_{\bar{i}}\Phi - i(T_a)_{i0}\partial_i\Phi - i(T_a)_{ij} z^j * \partial_i \Phi\right]. \tag{3.14}$$

Here T_a's are bases of the fundamental representation matrices of the Lie algebra $su(N+1)$ satisfying $[T_a, T_b] = i f_{abc} T_c$ with the structure constants f_{abc}, and their indices are assigned as

$$T_a = \left(\begin{array}{c|c} (T_a)_{00} & (T_a)_{0j} \\ \hline (T_a)_{i0} & (T_a)_{ij} \end{array}\right) \in su(N+1, \mathbb{C}).$$

These are represented by using the creation and annihilation operators as

$$\begin{aligned} P_a = -i \Big[&(T_a)_{00}(1-\hbar n) + \sqrt{\hbar}(T_a)_{0i} a_i^\dagger * (1-\hbar n)_*^{1/2} \\ &+ \sqrt{\hbar}(T_a)_{i0}(1-\hbar n)_*^{1/2} * a_i + \hbar (T_a)_{ij} a_j^\dagger * a_i \Big]. \end{aligned} \tag{3.15}$$

Here, we used

$$\partial_{\bar{i}}\Phi = \frac{z_i}{1+|z|^2} = z_i * (1-\tilde{n}) = \sqrt{\hbar} a_i^\dagger * (1-\hbar n)_*^{1/2}. \tag{3.16}$$

The Killing potentials constitute the Lie algebra $su(N+1)$,

$$[P_a, P_b]_* = -\hbar f_{abc} P_c. \tag{3.17}$$

In [18], it is shown that $e^{-\Phi/\hbar}$ corresponds to a vacuum state,

$$|\vec{0}\rangle\langle\vec{0}| = e^{-\Phi/\hbar}, \tag{3.18}$$

which satisfies

$$a_i * |\vec{0}\rangle\langle\vec{0}| = \frac{1}{\sqrt{\hbar}}(1-\tilde{n})_*^{-1/2} * \partial_i\Phi * e^{-\Phi/\hbar} = 0, \tag{3.19}$$

$$|\vec{0}\rangle\langle\vec{0}| * a_i^\dagger = \frac{1}{\sqrt{\hbar}} e^{-\Phi/\hbar} * z^i * (1-\tilde{n})_*^{1/2} = 0, \tag{3.20}$$

$$\Big(|\vec{0}\rangle\langle\vec{0}|\Big) * \Big(|\vec{0}\rangle\langle\vec{0}|\Big) = e^{-\Phi/\hbar} * e^{-\Phi/\hbar} = e^{-\Phi/\hbar} = |\vec{0}\rangle\langle\vec{0}|. \tag{3.21}$$

The bases of the set of linear operators acting on the Fock space are given by

$$|\vec{n}\rangle\langle\vec{m}| = |n_1, \cdots, n_N\rangle\langle m_1, \cdots, m_N|$$
$$= \frac{1}{\sqrt{n_1! \cdots n_N!}}(a_1^\dagger)_*^{n_1} * \cdots * (a_N^\dagger)_*^{n_N} * |\vec{0}\rangle\langle\vec{0}| * (a_1)_*^{m_1} * \cdots * (a_N)_*^{m_N} \frac{1}{\sqrt{m_1! \cdots m_N!}} \tag{3.22}$$

$$= \frac{1}{\sqrt{\prod_{i=1}^N n_i! m_i!}} \frac{\Gamma(1/\hbar+1)}{\sqrt{\Gamma(1/\hbar-|n|+1)\Gamma(1/\hbar-|m|+1)}} \frac{\prod_{j=1}^N (z_j)^{n_j}(\bar{z}_j)^{m_j}}{(1+|z|^2)^{\frac{1}{\hbar}}}, \tag{3.23}$$

where $(a)_*^n = \overbrace{a * \cdots * a}^{n}$ and $|m| = \sum_{i=1}^N m_i,\ |n| = \sum_{i=1}^N n_i$. Here we used

$$a_{i_1}^\dagger * \cdots * a_{i_k}^\dagger = \left(\frac{1}{\sqrt{\hbar}}\right)^k z^{i_1} * (1-\tilde{n})^{1/2} * \cdots * z^{i_k} * (1-\tilde{n})^{1/2}$$
$$= z^{i_1} * \cdots * z^{i_k} * (1/\hbar - n - (k-1))^{1/2} * (1/\hbar - n - (k-2))^{1/2} * \cdots * (1/\hbar - n)^{1/2}$$

and

$$a_{i_1}^\dagger * \cdots * a_{i_k}^\dagger * |\vec{0}\rangle\langle\vec{0}| = \sqrt{(1/\hbar)(1/\hbar-1)\cdots(1/\hbar-k+1)}\, z^{i_1} * \cdots * z^{i_k} * e^{-\Phi/\hbar}$$
$$= \sqrt{\frac{\Gamma(1/\hbar+1)}{\Gamma(1/\hbar-k+1)}} \frac{z^{i_1}\cdots z^{i_k}}{(1+|z|^2)^{1/\hbar}}. \tag{3.24}$$

We make a relation between the trace Tr on the Fock space and the integration on $\mathbb{C}P^N$. We normalize the trace as

$$\mathrm{Tr}\ |\vec{n}\rangle\langle\vec{m}| = \delta_{\vec{n},\vec{m}}. \tag{3.25}$$

Next, we calculate

$$I(\vec{n},\vec{m}) = \int |\vec{n}\rangle\langle\vec{m}|\ \sqrt{g} \prod_{i=1}^N dz^i d\bar{z}^i. \tag{3.26}$$

First, we have

$$
\begin{aligned}
I(\vec{n},\vec{m}) =& \frac{1}{\sqrt{\prod_{i=1}^{N} n_i! m_i!}} \frac{\Gamma(1/\hbar+1)}{\sqrt{\Gamma(1/\hbar-|n|+1)\Gamma(1/\hbar-|m|+1)}} \\
& \times \int \left(\prod_{i=1}^{N} dz^i d\bar{z}^i\right) \frac{\prod_{i=1}^{N}(z^i)^{n_i}(\bar{z}^i)^{m_i}}{(1+|z|^2)^{1/\hbar+N+1}}.
\end{aligned} \tag{3.27}
$$

By taking the parametrizations, $z^i = \sqrt{y_i}e^{i\theta_i}$, the integrations by the angle variables give $(2\pi)^N \delta_{\vec{n},\vec{m}}$. Then $I(\vec{n},\vec{m})$ becomes

$$
I(\vec{n},\vec{m}) = \delta_{\vec{n},\vec{m}} \frac{1}{\left(\prod_{i=1}^{N} n_i!\right)} \frac{(2\pi)^N \Gamma(1/\hbar+1)}{\Gamma(1/\hbar-|n|+1)} \int_0^\infty \left(\prod_{i=1}^{N} dy^i\right) \frac{\prod_{i=1}^{N} y_i^{n_i}}{\left(1+\sum_{i=1}^{N} y_i\right)^{1/\hbar+N+1}}. \tag{3.28}
$$

The y_i integrations are performed recursively from $i=1$ to $i=N$ by using the following equation,

$$
\begin{aligned}
&\int_0^\infty dy_i \frac{y_i^{n_i}}{\left(1+\sum_{j=i}^{N} y_j\right)^{1/\hbar+N+1-\sum_{j=1}^{i-1}(n_j+1)}} \\
&= \frac{1}{\left(1+\sum_{j=i+1}^{N} y_j\right)^{1/\hbar+N+1-\sum_{j=1}^{i}(n_j+1)}} \int_0^\infty dx \frac{x^{n_i}}{(1+x)^{1/\hbar+N+1-\sum_{j=1}^{i-1}(n_j+1)}} \\
&= \frac{n_i!\Gamma(1/\hbar+N+1-\sum_{j=1}^{i}(n_j+1))}{\Gamma(1/\hbar+N+1-\sum_{j=1}^{i-1}(n_j+1))} \frac{1}{\left(1+\sum_{j=i+1}^{N} y_j\right)^{1/\hbar+N+1-\sum_{j=1}^{i}(n_j+1)}},
\end{aligned} \tag{3.29}
$$

where x is defined as

$$
x = \frac{y_i}{1+\sum_{j=i+1}^{N} y_j}. \tag{3.30}
$$

Then we find

$$
I(\vec{n},\vec{m}) = \delta_{\vec{n},\vec{m}} c_N(\hbar), \tag{3.31}
$$

where

$$
c_N(\hbar) = \frac{(2\pi)^N \Gamma(1/\hbar+1)}{\Gamma(1/\hbar+N+1)}. \tag{3.32}
$$

The condition that all of the integrations converge is given by

$$
|n|, |m| < \frac{1}{\hbar} + 1. \tag{3.33}
$$

From (3.25) and (3.31), Tr on the Fock space is related to the integration,

$$\mathrm{Tr}\ |\vec{n}\rangle\langle\vec{m}| = \frac{1}{c_N}\int |\vec{n}\rangle\langle\vec{m}|\ \sqrt{g}\prod_{i=1}^{N} dz^i d\bar{z}^i. \tag{3.34}$$

When $1/\hbar$ is equal to a positive integer L, the space spanned by the bases $|\vec{n}\rangle\langle\vec{m}|$ is consistently restricted to those with $0 \le |n|, |m| \le L$. Then, it can be shown that the bases are complete,

$$\sum_{0\le|n|\le L} |\vec{n}\rangle\langle\vec{n}| = \sum_{0\le|n|\le L} \frac{L!}{(\prod_{i=1}^{N} n_i!)(L-|n|)!}\frac{\prod_{i=1}^{N}\left(|z^i|^2\right)^{n_i}}{(1+|z|^2)^L} = 1, \tag{3.35}$$

where $\sum_{0\le|n|\le L}$ denotes the summation over all partitions $\vec{n} = (n_1, \cdots, n_i, \cdots, n_N)$ satisfying $0 \le |n| = \sum_{i=1}^{N} n_i \le L$.

Let us consider $U(k)$ gauge theories in the Fock representation. We here take a gauge field being anti-hermitian, and a curvature being hermitian,

$$\mathcal{A}_a^\dagger = -\mathcal{A}_a, \qquad \mathcal{F}_{ab}^\dagger = \mathcal{F}_{ab}. \tag{3.36}$$

They are expressed in the Fock representation as

$$\mathcal{A}_a = i\sum_{\alpha,\vec{n},\vec{m}} \mathcal{A}^\alpha_{a;\vec{n},\vec{m}}\ t_\alpha\ |\vec{n}\rangle\langle\vec{m}|, \tag{3.37}$$

$$\mathcal{F}_{ab} = \sum_{\alpha,\vec{n},\vec{m}} \mathcal{F}^\alpha_{ab;\vec{n},\vec{m}}\ t_\alpha\ |\vec{n}\rangle\langle\vec{m}|, \tag{3.38}$$

where t_α $(\alpha = 1, 2, \ldots, k^2)$ are $d\times d$ hermitian matrices as a basis of a representation of the Lie algebra $u(k)$ and $\mathcal{A}^\alpha_{a;\vec{n},\vec{m}}, \mathcal{F}^\alpha_{ab;\vec{n},\vec{m}} \in \mathbb{C}$. The anti-hermiticity of $\mathcal{A}_a$ and the hermiticity of $\mathcal{F}_{ab}$ lead to

$$\mathcal{A}^\alpha_{a;\vec{n},\vec{m}} = \overline{\mathcal{A}^\alpha_{a;\vec{m},\vec{n}}}, \qquad \mathcal{F}^\alpha_{ab;\vec{n},\vec{m}} = \overline{\mathcal{F}^\alpha_{ab;\vec{m},\vec{n}}}. \tag{3.39}$$

An element of the $U(k)$ gauge transformation group $\mathcal{U}(k)$ is written as

$$U = \sum_{\Lambda,\vec{n},\vec{m}} U^\Lambda_{\vec{n},\vec{m}} M_\Lambda |\vec{n}\rangle\langle\vec{m}|, \tag{3.40}$$

where M_Λ are bases of $GL(d;\mathbb{C})$. From the unitarity, $U * U^\dagger = 1$, the following condition is imposed

$$\sum_{\Lambda,\Lambda',\vec{m}} M_\Lambda M^\dagger_{\Lambda'} U^\Lambda_{\vec{n},\vec{m}} \overline{U^{\Lambda'}_{\vec{n}',\vec{m}}} = 1_{d\times d}\delta_{\vec{n},\vec{n}'}. \tag{3.41}$$

In short, $\sum_\Lambda M_\Lambda \otimes U^\Lambda_{\vec{n},\vec{m}}$ is a unitary matrix.

In the Fock representation, the action functional of the gauge field is expressed as

$$S = \frac{c_N}{4} \mathrm{Tr}\ \mathrm{tr}\ \mathcal{F}_{ab}\mathcal{F}_{ab}, \tag{3.42}$$

where Tr is the trace on the Fock space and tr is the one for $d \times d$ matrices. Here, we used the Killing form for $\mathbb{C}P^N$, $\eta^{ab} = \delta^{ab}$. By using the Fock representation of the curvature (3.38) and the hermiticity condition of $\mathcal{F}_{ab}$ (3.39), we finally find

$$S = \frac{c_N}{4} \sum_{\alpha,\vec{n},\vec{m}} \left|\mathcal{F}^\alpha_{ab;\vec{n},\vec{m}}\right|^2. \tag{3.43}$$

The action for gauge fields (2.14) proposed in [14] is a formal power series in the noncommutative parameter. In the formal power series, one can not discuss whether the action is positive definite or not, and thus it is not possible to use the minimum action principle. To avoid this issue, the states was restricted within the Fock representation, and the positive action functional was obtained by the processes in this section.

At the end of this section, we comment about a relation between our models and the gauge theories on the fuzzy $\mathbb{C}P^N$. As a result of the restriction (3.33), the Hilbert space of our gauge theories becomes a finite dimensional space. So we expect our gauge theories are equivalent to some gauge theories on fuzzy $\mathbb{C}P^N$ [3, 7, 6].

4 New BPS-like equations in noncommutative $\mathbb{C}P^1$ and $\mathbb{C}P^2$

4.1 Equations of Motion and Bianchi Identities

Let us consider the action for gauge fields on $\mathbb{C}P^N$, (2.14) in the Fock representation. From a variation of the action with respective to the gauge field,

$$\delta S = \frac{1}{2}\int d^N z \sqrt{g}\ \mathrm{tr}\left(\mathcal{L}_a \delta\mathcal{A}_b - \mathcal{L}_b \delta\mathcal{A}_a - 2i[\delta\mathcal{A}_a, \mathcal{A}_b]_* - i f_{abc}\delta\mathcal{A}_c\right) * \mathcal{F}^{ab}, \tag{4.1}$$

the equations of motion are derived as

$$\mathcal{D}_b \mathcal{F}_{ab} - \frac{i}{2} f_{abc}\mathcal{F}_{bc} = 0. \tag{4.2}$$

$\mathcal{D}_a = \mathcal{L}_a - i[\mathcal{A}_a,\ \ ]_*$ is the covariant derivative for the adjoint representation. Note that the curvature is rewritten as

$$\mathcal{F}_{ab} = i[Q_a, Q_b]_* + f_{abc}Q_c, \tag{4.3}$$

where, using the Killing potential $\mathcal{P}_a = -\frac{i}{\hbar}P_a$, Q_a is defined as

$$Q_a = \mathcal{P}_a - i\mathcal{A}_a. \tag{4.4}$$

From the Jacobi identities,

$$[Q_a, [Q_b, Q_c]_*]_* + [Q_b, [Q_c, Q_a]_*]_* + [Q_c, [Q_a, Q_b]_*]_* = 0, \tag{4.5}$$

the Bianchi identities are derived as

$$\mathcal{D}_a\mathcal{F}_{bc} + if_{bcd}\mathcal{F}_{ad} + \mathcal{D}_b\mathcal{F}_{ca} + if_{cad}\mathcal{F}_{bd} + \mathcal{D}_c\mathcal{F}_{ab} + if_{abd}\mathcal{F}_{cd} = 0. \tag{4.6}$$

4.2 New BPS-like equations on noncommutative $\mathbb{C}P^1$

In this subsection we derive BPS-like equations on $\mathbb{C}P^1$. The isometry group of $\mathbb{C}P^1$ is $SU(2)$, which is a three-dimensional space. Hence, it seems that there is an analogy with gauge theories on $\mathbb{R}^3$. Let us introduce an adjoint scalar field ϕ which transforms under a gauge transformation as

$$\phi \to U^{-1} * \phi * U, \qquad U \in \mathcal{U}(k), \tag{4.7}$$

and its covariant derivative is given by

$$\mathcal{D}_a\phi = \mathcal{L}_a\phi - i[\mathcal{A}_a, \phi]_*. \tag{4.8}$$

Let us consider the following gauge invariant action functional with the gauge group $\mathcal{U}(k)$

$$S = \frac{1}{4}\int \mu_g \mathrm{tr}(\mathcal{F}_{ab} * \mathcal{F}_{ab} - 2\mathcal{D}_a\phi * \mathcal{D}_a\phi), \tag{4.9}$$

where the Killing form of $su(2)$, $\eta^{ab} = \delta^{ab}$, is used. In the Fock representation, the action is written as

$$S = \frac{c_1}{4}\mathrm{Tr}\ \mathrm{tr}(\mathcal{F}_{ab} * \mathcal{F}_{ab} - 2\mathcal{D}_a\phi * \mathcal{D}_a\phi). \tag{4.10}$$

As similar to the monopole theory of the Yang-Mills-Higgs model on $\mathbb{R}^3$, we can rewrite the above action as follows.

$$\frac{2}{c_1}S = \mathrm{Tr}\ \mathrm{tr}\left\{\left|(i\mathcal{D}_a\phi \pm \mathcal{B}_a)\right|^2 \mp i\mathcal{L}_a(\mathcal{B}_a\phi + \phi\mathcal{B}_a)\right\}, \tag{4.11}$$

where $\mathcal{B}_a$ is defined, using the structure constant of $su(2)$ $f_{abc} = \epsilon_{abc}$, as

$$\mathcal{B}_a = \frac{1}{2}\epsilon_{abc}\mathcal{F}_{bc}. \tag{4.12}$$

Here the following relations are used;

$$\mathcal{B}_a\mathcal{B}_a = \frac{1}{2}\mathcal{F}_{ab}\mathcal{F}_{ab}, \tag{4.13}$$

$$\mathcal{D}_a\mathcal{B}_a = \frac{1}{2}\epsilon_{abc}\left[Q_a, \mathcal{F}_{bc}\right]_* = \frac{1}{2}\epsilon_{abc}\left[Q_a, i\left[Q_b, Q_c\right]_* + \epsilon_{bcd}Q_d\right]_* = 0. \tag{4.14}$$

The second term in (4.11) ,

$$\int \mu_g \mathrm{tr}\ \left\{i\mathcal{L}_a(\mathcal{B}_a\phi + \phi\mathcal{B}_a)\right\}, \tag{4.15}$$

is a total divergence $i\int \partial_\mu\{\mu_g \text{ tr } \xi_a^\mu(B_a\phi+\phi B_a)\}$, in other words, topological charge. Finally, the BPS-like equations are obtained

$$\mathcal{B}_a \pm i\mathcal{D}_a\phi = 0. \tag{4.16}$$

The solutions of (4.16) satisfy the equations of motion of (4.9);

$$\mathcal{D}_a\mathcal{D}_a\phi = 0, \tag{4.17}$$

$$\mathcal{D}_b\mathcal{F}_{ab} - \frac{i}{2}\epsilon_{abc}\mathcal{F}_{bc} + i\left[\phi, \mathcal{D}_a\phi\right]_* = 0, \tag{4.18}$$

because

$$\mathcal{D}_a\mathcal{D}_a\phi = \pm i\mathcal{D}_a\mathcal{B}_a = 0$$

by the Bianchi identities (4.14), and

$$\begin{aligned}
\mathcal{D}_b\mathcal{F}_{ab} - \frac{i}{2}\epsilon_{abc}\mathcal{F}_{bc} &= \epsilon_{abc}\mathcal{D}_b\mathcal{B}_c - i\mathcal{B}_a \\
&= \mp\frac{i}{2}\epsilon_{abc}\left[\mathcal{D}_b, \mathcal{D}_c\right]_*\phi \mp \mathcal{D}_a\phi \\
&= \mp\left[\mathcal{B}_a, \phi\right]_* \\
&= -i\left[\phi, \mathcal{D}_a\phi\right]_*.
\end{aligned}$$

The BPS-like equations (4.16) are similar to Bogomolny's monopole equations in the Yang-Mills-Higgs model in $\mathbb{R}^3$, and the derivation processes are parallel with them [2]. The similarities arise from the following facts: $SU(2)$ and $\mathbb{R}^3$ are three-dimensional spaces, and the Killing form of $SU(2)$ used in the action (4.9) is the same as the Euclidean metric of $\mathbb{R}^3$. However the equations (4.16) are completely different from the Bogomolny's monopole equations. It can be seen from the following example.

For simplicity, we consider the $U(2)$ gauge theory on noncommutative $\mathbb{C}P^1$. In [1], Aoki, Iso and Nagao constructed a 't Hooft-Polyakov monopole on an Fuzzy sphere. Their solution is given by

$$\mathcal{A}_a = it_a, \tag{4.19}$$

where t_a is a some generator of the Lie algebra corresponding to $SU(2)$ which is common to both the gauge group and the isometry group. The curvature of this solution vanishes i.e. $\mathcal{F}_{ab} = 0$, but this solution is not trivial. (See also (4.29) in the next subsection.) Indeed it is shown that the solution has the monopole charge -1 in [1]. (Note that the monopole charge in [1] is different from the topological charge (4.15).) This monopole solution is included in solutions of the above BPS-like equations (4.16). Because $\mathcal{B}_a = \frac{1}{2}\epsilon_{abc}\mathcal{F}_{bc} = 0$, if we choose $\phi = 0$, then $\mathcal{A}_a = it_a$ satisfies (4.16), too. Hence, we find this configuration is a nontrivial solution of (4.16).

Some questions about the BPS-like equations (4.16) arise naturally. Are there solutions with non-zero $\mathcal{B}_a$? Are there any other solutions with non-zero topological charge (4.15)? Can we solve the BPS-like equations (4.16) systematically? These problems are left open.

4.3 New BPS-like equations on noncommutative $\mathbb{C}P^2$

In Section 4.2, we found that there is an analogy between a gauge theory on the three-dimensional Euclidean space and our gauge theory on noncommutative $\mathbb{C}P^1$, because the dimension of the isometry group $SU(2)$ of noncommutative $\mathbb{C}P^1$ is three and the Killing form of $su(2)$ plays the role of the Euclidean metric. It is natural to expect that this analogy extend to the case of noncommutative $\mathbb{C}P^2$. Since the dimension of the isometry group $SU(3)$ of $\mathbb{C}P^2$ is eight and the Killing form of the Lie algebra $su(2)$ is δ_{ab}, we draw an analogy with a gauge theory on $\mathbb{R}^8$. Similarly to the generalized instanton equations on the $\mathbb{R}^8$ given by Corrigan et al. [4], we try to derive new BPS equations on $\mathbb{C}P^2$ in this subsection.

We introduce T_{abcd} $(a, b, c, d = 1, \dots, 8)$ which is completely anti-symmetric with respect to the indices a, b, c, d. At first, we consider the case that T_{abcd} is a constant, that is, $\partial_i T_{abcd} = \partial_{\bar{i}} T_{abcd} = 0$. Let us put conditions like the instanton equations,

$$T_{abcd}\mathcal{F}_{cd} = 2\lambda\mathcal{F}_{cd}, \tag{4.20}$$

where λ is a non-zero constant. The consistency of the conditions requires that $T_{abcd}\,'s$ need to satisfy

$$T_{abcd}T_{cdef} = 2\lambda^2\left(\delta_{ae}\delta_{bf} - \delta_{af}\delta_{be}\right). \tag{4.21}$$

From the Bianchi identities (4.6), we obtain

$$T_{abcd}\left(\mathcal{D}_b\mathcal{F}_{cd} + if_{bce}\mathcal{F}_{de}\right) = 0. \tag{4.22}$$

Using these conditions (4.20), these equations become

$$\mathcal{D}_b\mathcal{F}_{ab} + \frac{i}{2\lambda}T_{abcd}f_{bce}\mathcal{F}_{de} = 0. \tag{4.23}$$

Comparing these equations with the equations of motion (4.2), further conditions have to be imposed,

$$T_{abcd}f_{bce}\mathcal{F}_{de} = -\lambda f_{abc}\mathcal{F}_{bc}. \tag{4.24}$$

Hence, we find new BPS-like equations by the combinations of (4.20) and (4.24) for constants T_{abcd}.

As similar to the case of $\mathbb{C}P^1$, we can construct a solution whose curvature $\mathcal{F}_{ab}$ vanishes. Consider the case of $SU(3)$ gauge theory on noncommutative $\mathbb{C}P^2$. Since the curvature is written as (4.3), it is easily seen that for bases t_a in a representation of $su(3)$ a gauge field

$$\mathcal{A}_a = it_a \tag{4.25}$$

is a configuration of $\mathcal{F}_{ab} = 0$ and this gauge connection is a solution of (4.20) and (4.24).

To observe that this solution gives nontrivial configurations of the gauge fields, we consider the commutative limit of it. For simplicity, we take the bases of the fundamental representation as t_a in (4.25), that is, t_a is equal to T_a in the Killing potential (3.14). In the commutative limit, the ordinary gauge fields, A_i and $A_{\bar{i}}$, are derived from $\mathcal{A}_a$ as follows;

$$A_i = -g_{i\bar{j}}\zeta_a^{\bar{j}}\mathcal{A}_a = -i(\partial_i P_a)\mathcal{A}_a = (\partial_i P_a)T_a, \tag{4.26}$$

$$A_{\bar{i}} = -g_{\bar{i}j}\zeta_a^{j}\mathcal{A}_a = i(\partial_{\bar{i}} P_a)\mathcal{A}_a = -(\partial_{\bar{i}} P_a)T_a. \tag{4.27}$$

The detailed relations between A_i, $A_{\bar{i}}$ and $\mathcal{A}_a$ are given in [14]. After a straightforward calculation, we obtain the curvature $F = dA + A\wedge A$ as

$$F_{ij} = F_{\bar{i}\bar{j}} = 0 \tag{4.28}$$

$$F_{i\bar{j}} = i\left(\begin{array}{c|c} -\frac{1}{1+|z|^2}(g_{i\bar{j}} - \bar{z}^i z^j) & \partial_l g_{i\bar{j}} \\ \hline \partial_{\bar{k}} g_{i\bar{j}} & -\frac{z^k\bar{z}^l g_{i\bar{j}}}{1+|z|^2} + g_{i\bar{k}}g_{l\bar{j}}(1+|z|^2) \end{array}\right). \tag{4.29}$$

Here, we represented a matrix $K = (K_{AB}) \in M_{N+1}(\mathbb{C})$, $(A, B = 0, 1, \ldots, N)$ as

$$K = \left(\begin{array}{c|c} K_{00} & K_{0l} \\ \hline K_{k0} & K_{kl} \end{array}\right), \tag{4.30}$$

with $k, l = 1, 2, \ldots, N$. Hence, the ordinary curvature F does not vanish though $\mathcal{F}_{ab} = 0$.

The above discussion about the flat connections $\mathcal{F}_{ab} = 0$ is valid for noncommutative gauge theories on $\mathbb{C}P^N$ for any N. When the isometry group $SU(N+1)$ of $\mathbb{C}P^N$ is a subgroup of gauge group, $\mathcal{A}_a = it_a$ is a flat connection in the meaning of $\mathcal{F}_{ab} = 0$, where t_a is a generator of the Lie algebra of a subgroup $SU(N+1)$ of the gauge group. In a case that t_a is in the fundamental representation, the ordinary curvature F has the form of (4.29) in the commutative limit and it does not vanish.

4.4 Another type of first order differential equations

Next, we should like to investigate another possibility for T_{abcd} in (4.20). In the following discussion, we consider only the commutative $\mathbb{C}P^N$. So far, we do not succeed in applying the following formulation to noncommutative $\mathbb{C}P^N$.

Since $\mathbb{C}P^N$ is the coset space $SU(N+1)/S(U(1) \times U(N))$, let us consider $\mathbb{C}P^N$ embedded in $SU(N+1)$. Projection operators h_{ab} to the tangential directions of $\mathbb{C}P^N$ in $SU(N+1)$ are constructed from the Killing vectors ζ_a^μ of $\mathbb{C}P^N$ as

$$h_{ab} = g_{\mu\nu}\zeta_a^\mu \zeta_b^\nu, \tag{4.31}$$

$$h_{ac}h_{cb} = h_{ab}. \tag{4.32}$$

From h_{ab} and the completely antisymmetric covariant tensor $E_{\mu\nu\rho\sigma}$ on $\mathbb{C}P^N$, we introduce two tensors J_{abcd} and T_{abcd} as follows;

$$J_{ab,cd} = \frac{1}{4}(h_{ac}h_{bd} - h_{ad}h_{bc}), \tag{4.33}$$

$$T_{ab,cd} = \frac{1}{4}\zeta_a^\mu \zeta_b^\nu \zeta_c^\rho \zeta_d^\sigma E_{\mu\nu\rho\sigma}. \tag{4.34}$$

Our definition of $E_{\mu\nu\rho\sigma}$ is $E_{1234} = \sqrt{g}$. These tensors satisfy the following relations,

$$J_{ab,ef}J_{ef,cd} = \frac{1}{2}J_{ab,cd}, \qquad T_{ab,ef}T_{ef,cd} = \frac{1}{2}J_{ab,cd}, \tag{4.35}$$

$$J_{ab,ef}T_{ef,cd} = T_{ab,ef}J_{ef,cd} = \frac{1}{2}T_{ab,cd}. \tag{4.36}$$

We then define orthogonal projection operators,

$$P^{(\pm)}_{ab,cd} = J_{ab,cd} \pm T_{ab,cd}, \tag{4.37}$$

$$P^{(\pm)}_{ab,cd}P^{(\pm)}_{cd,ef} = P^{(\pm)}_{ab,ef}, \qquad P^{(\pm)}_{ab,cd}P^{(\mp)}_{cd,ef} = 0. \tag{4.38}$$

Let us consider the following conditions for the curvature $\mathcal{F}_{ab}$,

$$P^{(-)}_{ab,cd}\mathcal{F}_{cd} = 0, \tag{4.39}$$

$$\left(I_{ab,cd} - P^{(+)}_{ab,cd}\right)\mathcal{F}_{cd} = 0, \tag{4.40}$$

where $I_{ab,cd} = \frac{1}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$. These are equivalent to

$$J_{ab,cd}\mathcal{F}_{cd} = \frac{1}{2}\mathcal{F}_{ab}, \tag{4.41}$$

$$T_{ab,cd}\mathcal{F}_{cd} = \frac{1}{2}\mathcal{F}_{ab}. \tag{4.42}$$

Combining these conditions and the Bianchi identities (4.6), we find

$$\begin{aligned} 0 &= T_{da,bc}(\mathcal{D}_a\mathcal{F}_{bc} + if_{bce}\mathcal{F}_{ae}) \\ &= \mathcal{D}_a(T_{da,bc}\mathcal{F}_{bc}) - (\mathcal{D}_aT_{da,bc})\mathcal{F}_{bc} + iT_{da,bc}f_{bce}\mathcal{F}_{ae} \\ &= \frac{1}{2}\mathcal{D}_a\mathcal{F}_{da} - (\mathcal{L}_aT_{da,bc})\mathcal{F}_{bc} + iT_{da,bc}f_{bce}\mathcal{F}_{ae}. \end{aligned} \tag{4.43}$$

The second and third terms in the rightest hand side of the above equation are calculated as

$$-(\mathcal{L}_aT_{da,bc})\mathcal{F}_{bc} = -\frac{i}{2}f_{ade}\mathcal{F}_{ea} - 2iT_{da,bc}f_{bce}\mathcal{F}_{ae}, \tag{4.44}$$

$$\begin{aligned} T_{da,bc}f_{bce}\mathcal{F}_{ae} &= 2T_{da,bc}f_{bce}T_{ae,fg}\mathcal{F}_{fg} \\ &= -\frac{1}{4}f_{bce}h_{de}h_{bf}h_{ch}\mathcal{F}_{fh} \\ &= -\frac{1}{4}f_{bce}h_{de}\mathcal{F}_{bc}. \end{aligned} \tag{4.45}$$

The proofs of (4.44) and (4.45) are given in Appendix A.

In Appendix B, the proof of the formulas,

$$f_{abc}\zeta^\mu_a\zeta^\nu_b\zeta^\rho_c = 0, \tag{4.46}$$

are given in the case of $\mathbb{C}P^N$. Using (4.41) and (4.46), the term in the rightest hand side of (4.45) vanishes

$$f_{bce}h_{de}\mathcal{F}_{bc} = f_{bce}h_{de}h_{bf}h_{ch}\mathcal{F}_{fh} = 0. \tag{4.47}$$

The equations (4.43) finally become

$$\mathcal{D}_b\mathcal{F}_{ab} - if_{abc}\mathcal{F}_{bc} = 0. \tag{4.48}$$

These equations are not same as the equations of motion (4.2). However, one can find a new action functional whose equations of motion are identified with (4.48).

In Ref. [13], the Chern-Simons like actions on coset spaces are provided. In our formulation, the corresponding action is written as

$$
\begin{aligned}
S_{CS} &= \int \mu_g f_{abc} \mathrm{tr} \left(\frac{i}{2} \mathcal{A}_a * \mathcal{L}_b \mathcal{A}_c + \frac{1}{3} \mathcal{A}_a * \mathcal{A}_b * \mathcal{A}_c + \frac{1}{4} f_{abd} \mathcal{A}_c * \mathcal{A}_d \right) \\
&= \int \mu_g f_{abc} \mathrm{tr} \left(\frac{i}{4} \mathcal{A}_a * \mathcal{F}_{bc} - \frac{1}{6} \mathcal{A}_a * \mathcal{A}_b * \mathcal{A}_c \right). \qquad (4.49)
\end{aligned}
$$

The equations of motion of S_{CS} are obtained as

$$
f_{abc} \mathcal{F}_{bc} = 0. \qquad (4.50)
$$

Under the gauge transformation (2.10), this action transforms as

$$
\begin{aligned}
S_{CS} \ \longrightarrow \ S_{CS} &+ \frac{1}{2} \int \mu_G \mathcal{L}_a \left(f_{abc} \mathrm{tr} \ \mathcal{L}_b U * U^{-1} * \mathcal{A}_a \right) \\
&+ \frac{i}{6} \int \mu_G f_{abc} \mathrm{tr} \ U^{-1} * \mathcal{L}_a U * U^{-1} * \mathcal{L}_b U * U^{-1} * \mathcal{L}_c U. \qquad (4.51)
\end{aligned}
$$

The second term on the right hand side is a total divergence, and thus it vanishes on a compact manifold without boundary. Furthermore, if we use the cyclic symmetry of the integration and trace, and the fact that the Killing vectors can be written as a star commutator, $\mathcal{L}_a = -\frac{i}{\hbar}[P_a, \ ]_*$, it can be shown that the third term also vanishes. Hence, S_{CS} is gauge invariant. Remark that this discussion can be applied to not only $\mathbb{C}P^N$ but also any homogeneous Kähler manifolds.

Let us consider an action which is a linear combination of the Yang-Mills type action and the Chern-Simons type action,

$$
S_{YM+CS} = S_{YM} + \alpha S_{CS}, \qquad (4.52)
$$

with a real constant parameter α. Then, the equations of motion of the action are given as

$$
\mathcal{D}_b \mathcal{F}_{ab} + \frac{i}{2}(\alpha - 1) f_{abc} \mathcal{F}_{bc} = 0. \qquad (4.53)
$$

Therefore, we can change the coefficient of $f_{abc}\mathcal{F}_{bc}$ in the equations of motion. If we put $\alpha = -1$, then the equations (4.53) are equal to (4.48).

At the end of this section, we make two comments. First, the equations (4.39) include usual instanton equation $*F = \pm F$ on $\mathbb{C}P^2$ in the tangent direction of $\mathbb{C}P^2$. The second comment is for the realization of this formulation in noncommutative $\mathbb{C}P^2$. To realize it, we have to construct the noncommutative version of the projection operators (4.37), (4.38). They have not been constructed until now.

5 Conclusions

In this article, using the Fock representation, we investigated gauge theories on noncommutative $\mathbb{C}P^N$ which is constructed by means of the deformation quantization with separation of variables. By virtue of the Fock representation, the minimal action principle makes sense in our gauge theories. We derived equations of motion and new BPS-like equations for $\mathbb{C}P^1$ and $\mathbb{C}P^2$.

It is found that there are analogies between the gauge theories on $\mathbb{C}P^N$ and gauge theories on $\mathbb{R}^{N^2+2N}$. Thus, the BPS-like equations on $\mathbb{C}P^1$ are similar to the monopole equations in the Yang-Mills-Higgs model on $\mathbb{R}^3$, and the BPS-like equations on $\mathbb{C}P^2$ are similar to the instanton equations on $\mathbb{R}^8$. We discussed some solutions for the BPS-like equations corresponding to $\mathcal{F}_{ab} = 0$, where we note that the vanishing of the curvature $\mathcal{F}_{ab}$ does not mean that the solutions are trivial, as we saw in Section 4. But we could not find new solutions with $\mathcal{F}_{ab} \neq 0$.

At the end of this article, we itemize problems being left unsolved.

1. How can we solve the new BPS-like equations systematically?
2. Are there solutions of the BPS-like equations with $\mathcal{F}_{ab} \neq 0$?
3. How can we characterize solutions of the BPS-like equations? Is there any topological invariant to characterize and classify solutions?

The problem 3 is deeply related with the quantum corrections to the topological term caused under the noncommutative deformation. We set the discussion of this problem aside for another day.

Acknowledgments
A.S. was supported in part by JSPS KAKENHI Grant Number 23540117.

A Proof of (4.44) and (4.45)

We let δ_a denote the Lie derivative corresponding to the Killing vector ζ_a^μ. The Lie derivatives of the Killing vector and $E_{\mu\nu\rho\sigma}$ are

$$\delta_a \zeta_b^\mu = \zeta_a^\nu \partial_\nu \zeta_b^\mu - \partial_\nu \zeta_a^\mu \zeta_b^\nu = i f_{abc} \zeta_c^\mu, \tag{A.1}$$

$$\delta_a E_{\mu\nu\rho\sigma} = 0, \tag{A.2}$$

where the Killing equation is used. $T_{ab,cd}$ is a scalar under the coordinate transformations, and thus

$$\delta_a T_{bc,de} = \zeta_a^\mu \partial_\mu T_{bc.de} = \mathcal{L}_a T_{bc,de}. \tag{A.3}$$

Using this, (4.44) is obtained as follows;

$$
\begin{aligned}
-(\mathcal{L}_a T_{da,bc})\mathcal{F}_{bc} &= -\frac{1}{4}\left[(\delta_a\zeta_d^\mu)\zeta_a^\nu\zeta_b^\rho\zeta_c^\sigma E_{\mu\nu\rho\sigma} + \zeta_d^\mu(\delta_a\zeta_a^\nu)\zeta_b^\rho\zeta_c^\sigma E_{\mu\nu\rho\sigma} + 2\zeta_d^\mu\zeta_a^\nu(\delta_a\zeta_b^\rho)\zeta_c^\sigma E_{\mu\nu\rho\sigma}\right. \\
&\quad \left. +\zeta_d^\mu\zeta_a^\nu\zeta_b^\rho\zeta_c^\sigma(\delta_a E_{\mu\nu\rho\sigma})\right]\mathcal{F}_{bc} \\
&= -\frac{1}{4}\left[if_{ade}\zeta_e^\mu\zeta_a^\nu\zeta_b^\rho\zeta_c^\sigma E_{\mu\nu\rho\sigma} + 2if_{abe}\zeta_d^\mu\zeta_a^\nu\zeta_e^\rho\zeta_c^\sigma E_{\mu\nu\rho\sigma}\right]\mathcal{F}_{bc} \\
&= (-if_{ade}T_{ea,bc} - 2if_{abe}T_{da,ec})\mathcal{F}_{bc} \\
&= -\frac{i}{2}f_{ade}\mathcal{F}_{ea} - 2iT_{da,bc}f_{bce}\mathcal{F}_{ae}.
\end{aligned}
\tag{A.4}
$$

(4.45) is also shown by the following calculations.

$$
\begin{aligned}
T_{da,bc}f_{bce}\mathcal{F}_{ae} &= 2T_{da,bc}f_{bce}T_{ae,fg}\mathcal{F}_{fg} \\
&= \frac{1}{8}f_{bce}\mathcal{F}_{fg}\zeta_d^\mu\zeta_a^\nu\zeta_b^\rho\zeta_c^\sigma E_{\mu\nu\rho\sigma}\zeta_a^{\mu'}\zeta_e^{\nu'}\zeta_f^{\rho'}\zeta_g^{\sigma'}E_{\mu'\nu'\rho'\sigma'} \\
&= -\frac{1}{8}f_{bce}\mathcal{F}_{fg}\zeta_d^\mu\zeta_b^\nu\zeta_c^\rho\zeta_e^{\mu'}\zeta_f^{\nu'}\zeta_g^{\rho'}E_{\mu\nu\rho\lambda}E_{\mu'\nu'\rho'}{}^{\lambda} \\
&= -\frac{1}{4}f_{bce}\mathcal{F}_{fg}\zeta_d^\mu\zeta_b^\nu\zeta_c^\rho\zeta_e^{\mu'}\zeta_f^{\nu'}\zeta_g^{\rho'}(g_{\mu\mu'}g_{\nu\nu'}g_{\rho\rho'} + g_{\mu\nu'}g_{\nu\rho'}g_{\rho\mu'} + g_{\mu\rho'}g_{\nu\mu'}g_{\rho\nu'}) \\
&= -\frac{1}{4}f_{bce}h_{de}\mathcal{F}_{bc}.
\end{aligned}
\tag{A.5}
$$

B Proof of (4.46)

Here, the notation for matrices in (4.30) is used. We define a $(N+1)\times(N+1)$ matrix P from the Killing potentials P_a of $\mathbb{C}P^N$ as

$$
P = P_a T_a,
\tag{B.1}
$$

where T_a is a generator of $su(N+1)$ in the fundamental representation. T_a is normalized so that

$$
(T_a)_{AB}(T_a)_{CD} = \delta_{AD}\delta_{BC} - \frac{1}{N+1}\delta_{AB}\delta_{CD}.
\tag{B.2}
$$

Then, P is written as

$$
P = \frac{-i}{1+|z|^2}\begin{pmatrix} 1 & \bar{z}^l \\ z^k & z^k\bar{z}^l \end{pmatrix} + \frac{i}{N+1}1_{N+1}.
\tag{B.3}
$$

The following relations hold for P and derivatives of P;

$$P\partial_i P = \frac{i}{N+1}\partial_i P, \qquad \partial_i P P = -i\frac{N}{N+1}\partial_i P, \tag{B.4}$$

$$P\partial_{\bar{i}} P = -i\frac{N}{N+1}\partial_{\bar{i}} P, \qquad \partial_{\bar{i}} P P = \frac{i}{N+1}\partial_{\bar{i}} P, \tag{B.5}$$

$$\partial_i P \partial_j P = \partial_{\bar{i}} P \partial_{\bar{j}} P = 0, \tag{B.6}$$

$$\mathrm{Tr}\ \partial_i P = \mathrm{Tr}\ \partial_{\bar{i}} P = 0, \qquad \mathrm{Tr}\ \partial_i P \partial_{\bar{j}} P = -g_{i\bar{j}}. \tag{B.7}$$

The Killing vectors of $\mathbb{C}P^N$ are represented by $\partial_i P$ and $\partial_{\bar{i}} P$ as

$$[T_a,\ P] = -\zeta_a^i \partial_i P - \zeta_a^{\bar{i}} \partial_{\bar{i}} P, \tag{B.8}$$

$$\zeta_a^i = -i g^{i\bar{j}} \mathrm{Tr}\ T_a \partial_{\bar{j}} P, \qquad \zeta_a^{\bar{i}} = i g^{\bar{i}j} \mathrm{Tr}\ T_a \partial_j P. \tag{B.9}$$

Using above relations, let us calculate $f_{abc}\zeta_a^{\bar{i}}\zeta_b^{\bar{j}}\zeta_c^{\bar{k}}$;

$$\begin{aligned} f_{abc}\zeta_a^{\bar{i}}\zeta_b^{\bar{j}}\zeta_c^{\bar{k}} &= -g^{\bar{i}l} g^{\bar{j}m} g^{\bar{k}n} \left(\mathrm{Tr}\ T_a[T_b, T_c]\right) \left(\mathrm{Tr}\ T_a \partial_l P\right) \left(\mathrm{Tr}\ T_b \partial_m P\right) \left(\mathrm{Tr}\ T_c \partial_n P\right) \\ &= -g^{\bar{i}l} g^{\bar{j}m} g^{\bar{k}n} \left(\mathrm{Tr}\ \left[T_c, \partial_l P\right] \partial_m P\right) \left(\mathrm{Tr}\ T_c \partial_n P\right) \\ &= 0. \end{aligned} \tag{B.10}$$

Here we used (B.2) in the second equality and (B.6) in the third equality. Similarly, $f_{abc}\zeta_a^{\bar{i}}\zeta_b^{\bar{j}}\zeta_c^k$, $f_{abc}\zeta_a^{\bar{i}}\zeta_b^j\zeta_c^k$ and $f_{abc}\zeta_a^i\zeta_b^j\zeta_c^k$ also vanish.

References

[1] H. Aoki, S. Iso and K. Nagao, "Ginsparg-Wilson relation and 't Hooft-Polyakov monopole on fuzzy 2 sphere," Nucl. Phys. B **684**, 162 (2004) [hep-th/0312199].

[2] E. B. Bogomolny, "Stability of Classical Solutions," Sov. J. Nucl. Phys. **24** (1976) 449 [Yad. Fiz. **24** (1976) 861].

[3] U. Carow-Watamura and S. Watamura, "Noncommutative geometry and gauge theory on fuzzy sphere," Commun. Math. Phys. **212**, 395 (2000) [hep-th/9801195].

[4] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, "First Order Equations for Gauge Fields in Spaces of Dimension Greater Than Four," Nucl. Phys. B **214**, 452 (1983).

[5] E. Corrigan and P. Goddard, "An n Monopole Solution With 4n-1 Degrees of Freedom," Commun. Math. Phys. **80**, 575 (1981).

[6] B. P. Dolan, I. Huet, S. Murray and D. O'Connor, "Noncommutative vector bundles over fuzzy CP^N and their covariant derivatives," JHEP **0707** (2007) 007 [hep-th/0611209].

[7] H. Grosse and H. Steinacker, "Finite gauge theory on fuzzy CP^2," Nucl. Phys. B **707**, 145 (2005) [hep-th/0407089].

[8] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, "A Large N reduced model as superstring," Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115].

[9] A. V. Karabegov, "On deformation quantization, on a Kahler manifold, associated to Berezin's quantization," Funct. Anal. Appl. **30**, 142 (1996).

[10] A. V. Karabegov, "Deformation quantizations with separation of variables on a Kahler manifold," Commun. Math. Phys. **180**, 745 (1996) [hep-th/9508013].

[11] A. V. Karabegov, "An explicit formula for a star product with separation of variables," [arXiv:1106.4112 [math.QA]].

[12] H. Kawai, S. Shimasaki and A. Tsuchiya, "Large N reduction on group manifolds," Int. J. Mod. Phys. A **25**, 3389 (2010) [arXiv:0912.1456 [hep-th]].

[13] H. Kawai, S. Shimasaki and A. Tsuchiya, "Large N reduction on coset spaces," Phys. Rev. D **81**, 085019 (2010) [arXiv:1002.2308 [hep-th]].

[14] Y. Maeda, A. Sako, T. Suzuki and H. Umetsu, "Gauge theories in noncommutative homogeneous Kähler manifolds," J. Math. Phys. **55**, 092301 (2014) [arXiv:1403.5727 [hep-th]].

[15] M. Müller-Bahns and N. Neumaier, "Invariant Star Products of Wick Type: Classification and Quantum Momentum Mappings," Lett. Math. Phys. **70**, 1 (2004).

[16] N. A. Nekrasov, "Trieste lectures on solitons in noncommutative gauge theories," [hep-th/0011095].

[17] A. Sako, " Recent developments in instantons in noncommutative $\mathbb{R}^4$," Adv. Math. Phys. **2010**(2010) , ID 270694, 28pp.

[18] A. Sako, T. Suzuki and H. Umetsu, "Explicit Formulas for Noncommutative Deformations of CP^N and CH^N," J. Math. Phys. **53**, 073502 (2012) [arXiv:1204.4030 [math-ph]].

[19] A. Sako, T. Suzuki and H. Umetsu, "Noncommutative CP^N and CH^N and their physics," J. Phys. Conf. Ser. **442**, 012052 (2013).

[20] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909**, 032 (1999) [hep-th/9908142].

[21] R. J. Szabo, "Quantum field theory on noncommutative spaces," Phys. Rept. **378**, 207 (2003) [hep-th/0109162].